Non-linearities and bias

Camille Bonvin

Kavli Institute for Cosmology and DAMTP
Cambridge

Saas Fee lectures Engelberg
March 2014

Summary

- Three important features of the power spectrum, not encoded in the original dark matter calculation.
- **Dark energy** dominates the universe today.
- We do not observe the correlation function in real space but in **redshift space**. The mapping from real to redshift space:
- Modifies the amplitude of the correlation function.
- Generates a dependence of the correlation function on the orientation of the pair: a **quadrupole** and an **hexadecapole**.
- We can extract these multipoles by weighting the average over the angles:

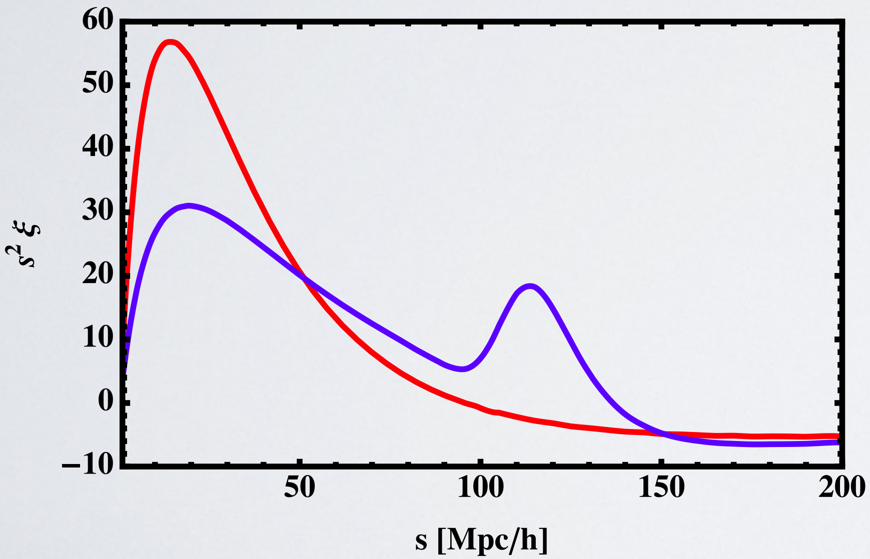
$$\xi_\ell = \frac{2\ell+1}{2}\int_{-1}^{1} d\mu\, \xi(r,s,\mu) P_\ell(\mu)$$

- This provides a measurement of the **growth rate** $f = \frac{a}{D_1}\frac{d}{da}D_1$

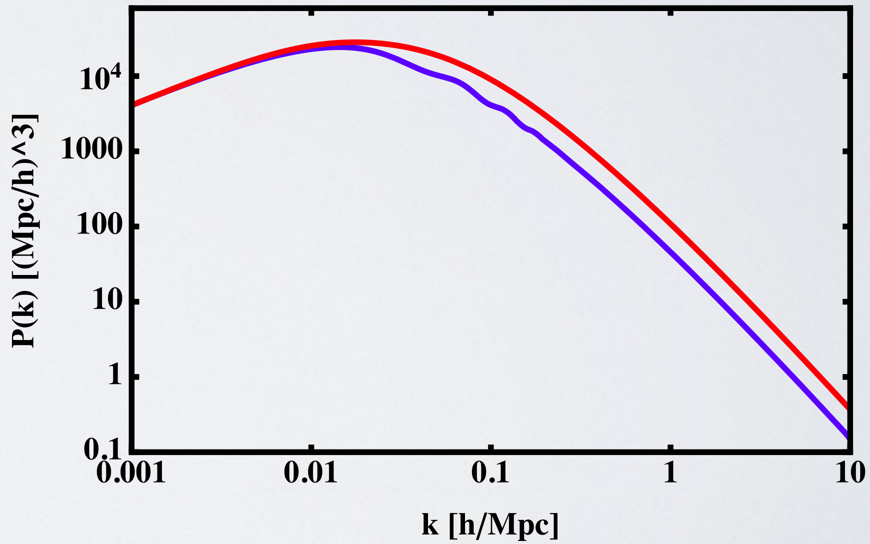
Summary

- **Baryon acoustic oscillations**: it is not correct to treat all the matter as dark matter.
- The subdominant baryons behave differently before recombination: they couple to the photons and oscillate.

Peak in $\xi(s)$



Wiggles in $P(k)$

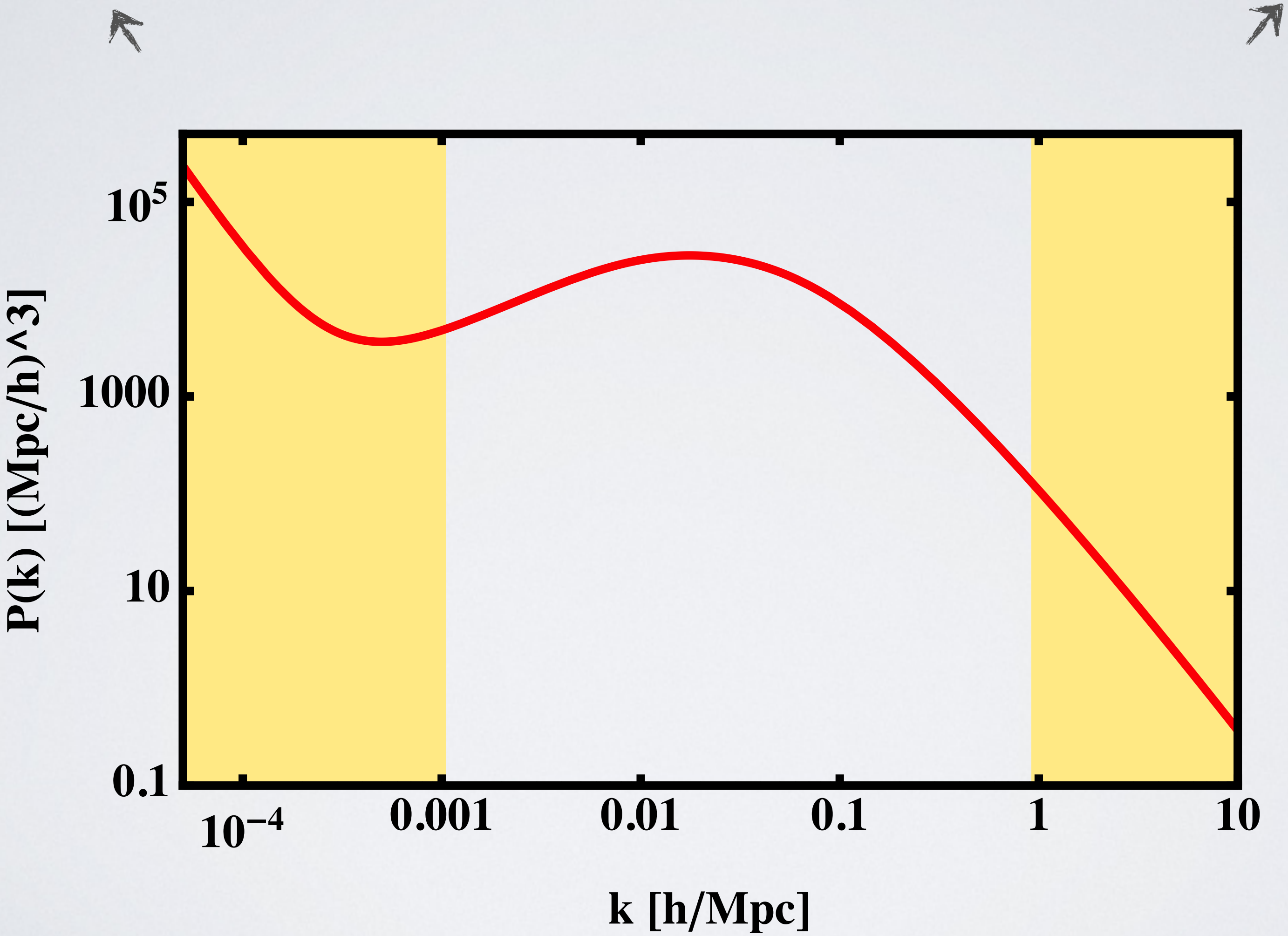


- The scale of the oscillations is used as a **standard ruler**.

Missing parts

large scales: corrections to the Newtonian calculation

small scales: corrections to the linear calculation



Non-linearities

- Basic equations: continuity and Euler.

$$\delta'_{dm}(\mathbf{x},\eta) + \nabla\Big[\big(1+\delta_{dm}(\mathbf{x},\eta)\big)\cdot\mathbf{v}_{dm}(\mathbf{x},\eta)\Big] = 0$$

$$\mathbf{v}'_{dm}(\mathbf{x},\eta) + \mathcal{H}\mathbf{v}_{dm}(\mathbf{x},\eta) + \mathbf{v}_{dm}(\mathbf{x},\eta)\cdot\nabla\mathbf{v}_{dm}(\mathbf{x},\eta) = -\nabla\Phi$$

- Linear regime: $\delta_{dm},\, v_{dm} \sim 10^{-3}$ → $\delta_{dm}\cdot v_{dm} \sim 10^{-6}$ **negligible**

$$\delta'_{dm}(\mathbf{x},\eta) + \nabla\mathbf{v}_{dm}(\mathbf{x},\eta) = 0$$

$$\mathbf{v}'_{dm}(\mathbf{x},\eta) + \mathcal{H}\mathbf{v}_{dm}(\mathbf{x},\eta) = -\nabla\Phi$$

- We know that $\delta_{dm} \ll 1$ is not always true, e.g. in **galaxies** and **clusters**.

- In this case the linear equations are wrong: $\delta_{dm}\cdot v_{dm} \gg \delta_{dm}$

Non-linearities

- Basic equations: continuity and Euler.

non-linear

$$\delta'_{dm}(\mathbf{x},\eta) + \nabla\Big[\big(1+\delta_{dm}(\mathbf{x},\eta)\big)\cdot\mathbf{v}_{dm}(\mathbf{x},\eta)\Big] = 0$$

$$\mathbf{v}'_{dm}(\mathbf{x},\eta) + \mathcal{H}\mathbf{v}_{dm}(\mathbf{x},\eta) + \mathbf{v}_{dm}(\mathbf{x},\eta)\cdot\nabla\mathbf{v}_{dm}(\mathbf{x},\eta) = -\nabla\Phi$$

- Linear regime: $\delta_{dm},\, v_{dm} \sim 10^{-3}$ → $\delta_{dm}\cdot v_{dm} \sim 10^{-6}$ **negligible**

$$\delta'_{dm}(\mathbf{x},\eta) + \nabla\mathbf{v}_{dm}(\mathbf{x},\eta) = 0$$

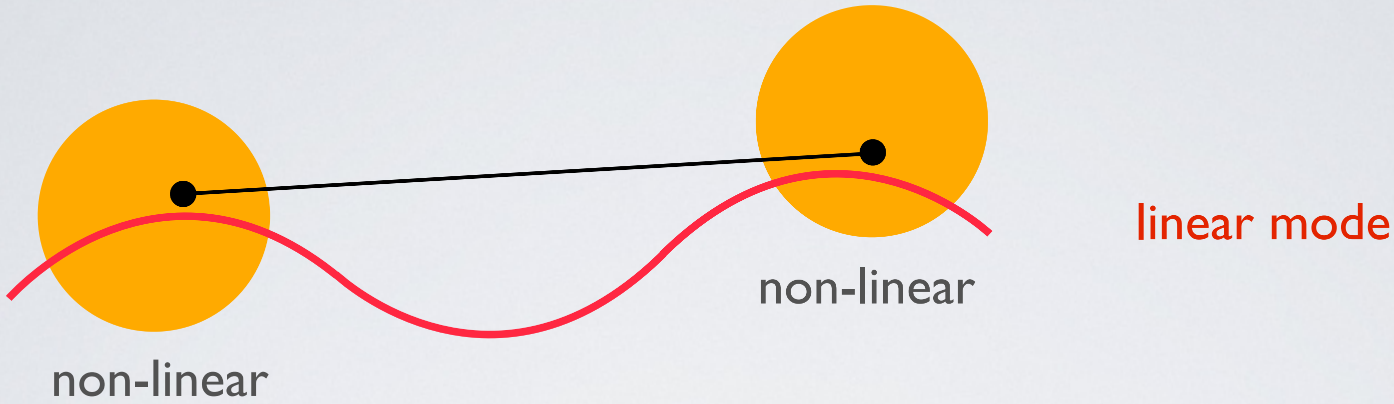
$$\mathbf{v}'_{dm}(\mathbf{x},\eta) + \mathcal{H}\mathbf{v}_{dm}(\mathbf{x},\eta) = -\nabla\Phi$$

- We know that $\delta_{dm} \ll 1$ is not always true, e.g. in **galaxies** and **clusters**.
- In this case the linear equations are wrong: $\delta_{dm}\cdot v_{dm} \gg \delta_{dm}$

Validity of the linear resolution

Is our linear calculation completely wrong?

The answer depends on the **scale**.



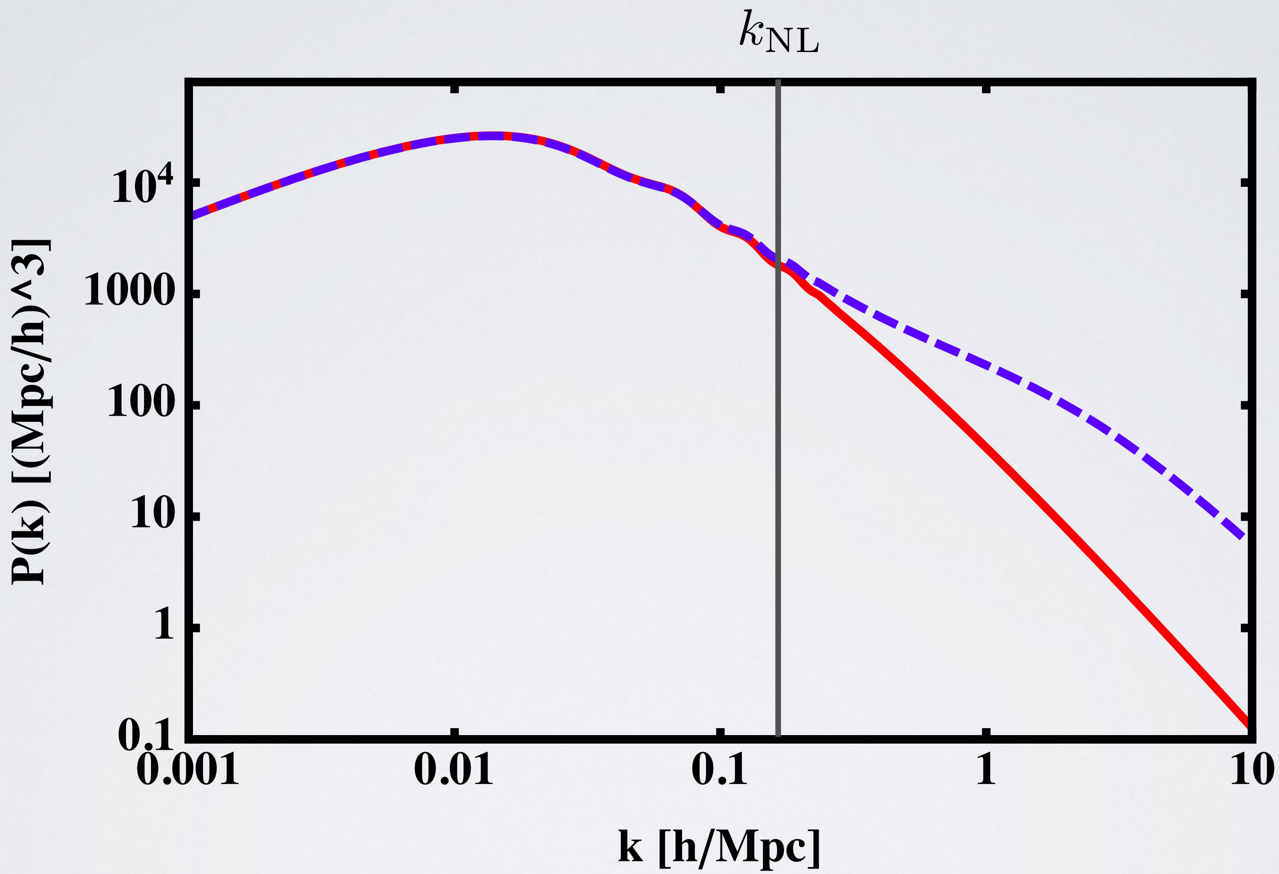
The non-linear physics is **not correlated**. Only the long wavelength modes induce correlations $\delta_{dm}(k_L, \eta) \ll 1$: linear physics is valid.

The galaxies share the same non-linear physics: they are **correlated** by $\delta_{dm}(k_{NL}, \eta) \gg 1$

→ Linear physics is not valid.

Validity of the linear resolution

The linear calculation is valid until $k_{\rm NL}$



What can we do in the **non-linear regime**?

Beyond the linear regime

$$\delta'_{dm}(\mathbf{x},\eta) + \nabla\Big[\big(1+\delta_{dm}(\mathbf{x},\eta)\big)\cdot\mathbf{v}_{dm}(\mathbf{x},\eta)\Big] = 0$$

$$\mathbf{v}'_{dm}(\mathbf{x},\eta) + \mathcal{H}\mathbf{v}_{dm}(\mathbf{x},\eta) + \mathbf{v}_{dm}(\mathbf{x},\eta)\cdot\nabla\mathbf{v}_{dm}(\mathbf{x},\eta) = -\nabla\Phi$$

- Without approximations, we cannot solve analytically these equations → numerical **N-body simulations**.
- The power spectrum can be measured from the simulations, but they take time and **depend** on the **cosmology**.
- An analytical description can help.
- One possibility is to do **perturbation theory**: keep terms up to a certain order in the equations.

Perturbation theory

- We expand: $$\delta_{dm} = \underset{\epsilon}{\delta_{dm}^{(1)}} + \underset{\epsilon^2}{\delta_{dm}^{(2)}} + \underset{\epsilon^3}{\delta_{dm}^{(3)}} + \dots$$
- If $\epsilon \ll 1$, the higher order terms are **negligible**: keep only $\delta_{dm}^{(1)}$
- If $\epsilon > 1$, the higher order terms are larger: keep the full expansion, i.e. **fully non-linear** resolution.
- If $\epsilon \lesssim 1$, we have a hierarchy $\delta_{dm}^{(3)} < \delta_{dm}^{(2)} < \delta_{dm}^{(1)}$, and we can **improve** the calculation by including **higher order** terms.
- Perturbation theory allows to calculate the power spectrum in the **mildly non-linear regime**, for a range of k.
- Beyond that we need something else.

Perturbation theory

We do a similar expansion for the **velocity**.

We use

$$\theta_{dm}(\mathbf{x},\eta) = \nabla \mathbf{v}_{dm}(\mathbf{x},\eta)$$

$$\theta_{dm}(\mathbf{k},\eta) = -k\, v_{dm}(\mathbf{k},\eta)$$

$$\theta_{dm},\ \delta \sim k^2\Phi$$

$$\theta_{dm} = \theta_{dm}^{(1)} + \theta_{dm}^{(2)} + \theta_{dm}^{(3)} + \dots$$

We insert the **expansions** into the continuity and Euler equations.

$$\delta'_{dm}(\mathbf{x},\eta) + \nabla\Big[\big(1+\delta_{dm}(\mathbf{x},\eta)\big)\cdot \mathbf{v}_{dm}(\mathbf{x},\eta)\Big] = 0$$

$$\mathbf{v}'_{dm}(\mathbf{x},\eta) + \mathcal{H}\mathbf{v}_{dm}(\mathbf{x},\eta) + \mathbf{v}_{dm}(\mathbf{x},\eta)\cdot\nabla\mathbf{v}_{dm}(\mathbf{x},\eta) = -\nabla\Phi$$

Linear order:

$$\delta_{dm}^{(1)'} + \theta_{dm}^{(1)} = 0$$

$$\theta_{dm}^{(1)'} + \mathcal{H}\theta_{dm}^{(1)} = k^2\Phi^{(1)} = -\frac{3}{2}\mathcal{H}^2\Omega_m(\eta)\delta_{dm}^{(1)}$$

Second order

$$\delta'_{dm}(\mathbf{x},\eta) + \nabla \mathbf{v}_{dm}(\mathbf{x},\eta) = -\nabla\big(\delta_{dm}(\mathbf{x},\eta)\cdot \mathbf{v}_{dm}(\mathbf{x},\eta)\big)$$

$$\delta^{(2)}_{dm} \qquad \theta^{(2)}_{dm} \qquad \delta^{(1)}_{dm}\cdot\theta^{(1)}_{dm} \qquad \epsilon^2$$

Fourier transform:

- left-hand side $\delta^{(2)'}_{dm} + \theta^{(2)}_{dm}$
- right-hand side $\int d^3\mathbf{x}\, e^{i\mathbf{k}\cdot\mathbf{x}}\, \nabla\big(\delta_{dm}(\mathbf{x},\eta)\mathbf{v}_{dm}(\mathbf{x},\eta)\big)$ (Fourier, Fourier)

the linear modes source
the second-order modes

$$\delta^{(2)'}_{dm}(\mathbf{k},\eta) + \theta^{(2)}_{dm}(\mathbf{k},\eta) =$$

$$-\int \frac{d^3\mathbf{k}_1}{(2\pi)^3}\int d^3\mathbf{k}_2\, \delta_D(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\frac{(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{k}_1}{k_1^2}\theta^{(1)}_{dm}(\mathbf{k}_1,\eta)\delta^{(1)}_{dm}(\mathbf{k}_2,\eta)$$

Second order

$$\delta'_{dm}(\mathbf{x},\eta)+\nabla\mathbf{v}_{dm}(\mathbf{x},\eta)=-\nabla\big(\delta_{dm}(\mathbf{x},\eta)\cdot\mathbf{v}_{dm}(\mathbf{x},\eta)\big)$$

$$\delta^{(2)}_{dm} \qquad \theta^{(2)}_{dm} \qquad \delta^{(1)}_{dm}\cdot\theta^{(1)}_{dm} \qquad \epsilon^2$$

Fourier transform:

- left-hand side $\int\frac{d^3\mathbf{k}_2}{(2\pi)^3}\,e^{-i\mathbf{k}_2\cdot\mathbf{x}}\,\delta_{dm}(\mathbf{k}_2,\eta)$ $\int\frac{d^3\mathbf{k}_1}{(2\pi)^3}\,e^{-i\mathbf{k}_1\cdot\mathbf{x}}\,\theta_{dm}(\mathbf{k}_1,\eta)$
- right-hand side $\int d^3\mathbf{x}\,e^{i\mathbf{k}\cdot\mathbf{x}}\,\nabla\big(\delta_{dm}(\mathbf{x},\eta)\mathbf{v}_{dm}(\mathbf{x},\eta)\big)$

the linear modes source
the second-order modes

$$\delta^{(2)'}_{dm}(\mathbf{k},\eta)+\theta^{(2)}_{dm}(\mathbf{k},\eta)=$$
$$-\int\frac{d^3\mathbf{k}_1}{(2\pi)^3}\int d^3\mathbf{k}_2\,\delta_D(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\frac{(\mathbf{k}_1+\mathbf{k}_2)\cdot\mathbf{k}_1}{k_1^2}\theta^{(1)}_{dm}(\mathbf{k}_1,\eta)\delta^{(1)}_{dm}(\mathbf{k}_2,\eta)$$

Solution

$$\theta^{(2)'}_{dm}(\mathbf{k},\eta) + \mathcal{H}\theta^{(2)}_{dm}(\mathbf{k},\eta) + \frac{3}{2}\Omega_m\mathcal{H}^2\delta^{(2)'}_{dm}(\mathbf{k},\eta) =$$

$$-\int\frac{d^3\mathbf{k}_1}{(2\pi)^3}\int d^3\mathbf{k}_2\,\delta_D(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)\frac{|\mathbf{k}_1+\mathbf{k}_2|^2\,\mathbf{k}_1\cdot\mathbf{k}_2}{2k_1^2k_2^2}\,\theta^{(1)}_{dm}(\mathbf{k}_1,\eta)\theta^{(1)}_{dm}(\mathbf{k}_2,\eta)$$

Coupled system of equations, with a known source term.

$$\theta^{(2)}_{dm}(\mathbf{k},\eta) = \int\frac{d^3\mathbf{k}_1 d^3\mathbf{k}_2}{(2\pi)^3}\delta_D(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)G_2(\mathbf{k}_1,\mathbf{k}_2,\eta)\delta^{(1)}_{dm}(\mathbf{k}_1,\eta)\delta^{(1)}_{dm}(\mathbf{k}_2,\eta)$$

$$\delta^{(2)}_{dm}(\mathbf{k},\eta) = \int\frac{d^3\mathbf{k}_1 d^3\mathbf{k}_2}{(2\pi)^3}\delta_D(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2)F_2(\mathbf{k}_1,\mathbf{k}_2,\eta)\delta^{(1)}_{dm}(\mathbf{k}_1,\eta)\delta^{(1)}_{dm}(\mathbf{k}_2,\eta)$$

The solution is:

$$\epsilon = \frac{3}{7}\left(\frac{\rho_m(\eta)}{\rho_{tot}(\eta)}\right)^{-\frac{1}{143}}$$

$$F_2 = \frac{1}{2}(1+\epsilon) + \frac{\hat{\mathbf{k}}_1\cdot\hat{\mathbf{k}}_2}{2}\left(\frac{\mathbf{k}_1}{\mathbf{k}_2}+\frac{\mathbf{k}_2}{\mathbf{k}_1}\right) + \frac{1}{2}(1-\epsilon)(\hat{\mathbf{k}}_1\cdot\hat{\mathbf{k}}_2)^2$$

Power spectrum

Equation for $\delta_{dm}^{(3)}$ and $\theta_{dm}^{(3)}$

$$\delta'_{dm}(\mathbf{x},\eta) + \nabla\mathbf{v}_{dm}(\mathbf{x},\eta) = -\nabla\big(\delta_{dm}(\mathbf{x},\eta)\cdot\mathbf{v}_{dm}(\mathbf{x},\eta)\big)$$

$$\delta_{dm}^{(3)} \qquad \theta_{dm}^{(3)} \qquad \delta_{dm}^{(2)}\,\theta_{dm}^{(1)} \qquad \delta_{dm}^{(1)}\,\theta_{dm}^{(2)} \qquad \epsilon^3$$

The second-order solutions source the third order, and so on.

Power spectrum

$$\langle\delta_{dm}\,\delta_{dm}\rangle = \langle\delta_{dm}^{(1)}\,\delta_{dm}^{(1)}\rangle + \langle\delta_{dm}^{(1)}\,\delta_{dm}^{(2)}\rangle + \langle\delta_{dm}^{(1)}\,\delta_{dm}^{(3)}\rangle + \langle\delta_{dm}^{(2)}\,\delta_{dm}^{(2)}\rangle$$

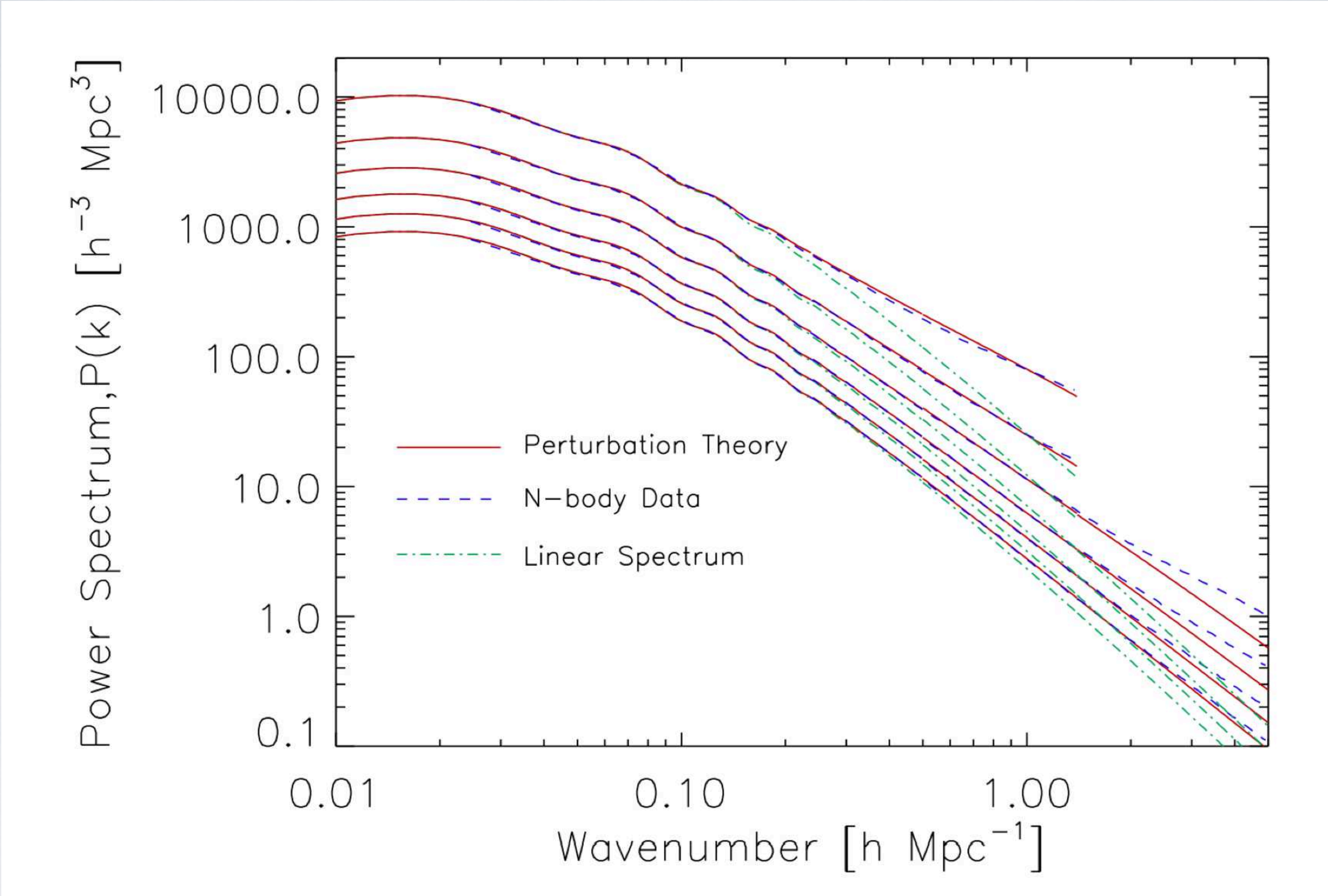
linear

$$\langle\delta_{dm}^{(1)}\,\delta_{dm}^{(1)}\delta_{dm}^{(1)}\rangle = 0$$

correction

Result

Credit: D. Jeong



At very small scales, the solution from perturbation theory is not good enough.

Can we do better analytically? Yes: with the Press and Schechter formalism.

Higher order correlations

In a Gaussian field $\langle\delta^{(1)}_{dm}\,\delta^{(1)}_{dm}\delta^{(1)}_{dm}\rangle = 0$ ➔ the power spectrum contains all the information.

Non-linearities generate **non-Gaussianities**:

$$\langle\delta_{dm}\,\delta_{dm}\,\delta_{dm}\rangle = \langle\delta^{(2)}_{dm}\,\delta^{(1)}_{dm}\,\delta^{(1)}_{dm}\rangle = \langle\delta^{(1)}_{dm}\,\delta^{(1)}_{dm}\,\delta^{(1)}_{dm}\,\delta^{(1)}_{dm}\rangle \neq 0$$

Even if the linear $\delta^{(1)}_{dm}$ remains Gaussian, the observed one is not.

The power spectrum is not sufficient to characterise δ_{dm}

Bispectrum:

$$\langle\delta_{dm}(\mathbf{k}_1,\eta)\delta_{dm}(\mathbf{k}_2,\eta)\delta_{dm}(\mathbf{k}_3,\eta)\rangle$$
$$= (2\pi)^3\delta_D(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)F_2(\mathbf{k}_1,\mathbf{k}_2,\eta)P(k_1,\eta)P(k_2,\eta) + \text{cyc.}$$

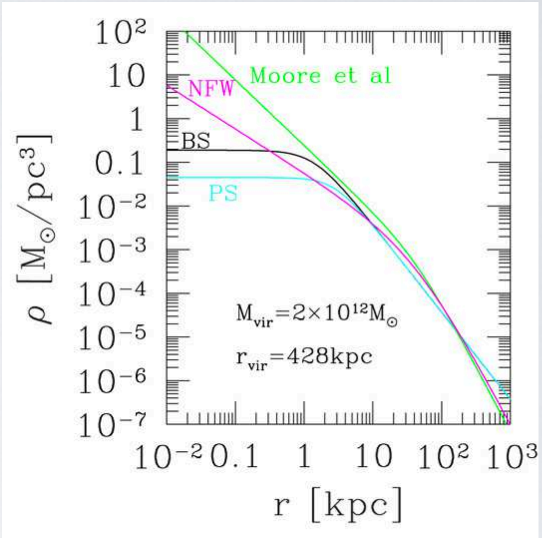
Press and Schechter formalism

How can we do better than perturbation theory?

Idea: describe the distribution of matter as a **collection of halos**.

halos with different masses

density profile



With such a description we can calculate the **correlation function:**

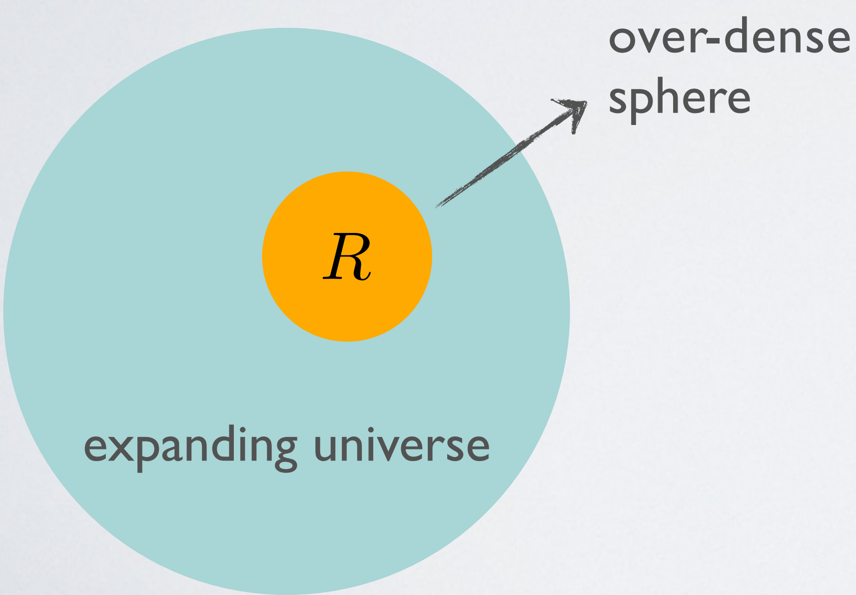
$$\rho(\mathbf{x}) = \sum_i M_i \cdot u(\mathbf{x}-\mathbf{x}_i, M_i)$$

$$\xi(\mathbf{x},\mathbf{x}') = \langle\rho(\mathbf{x})\rho(\mathbf{x}')\rangle - \langle\rho(\mathbf{x})\rangle\langle\rho(\mathbf{x}')\rangle$$

Spherical collapse

First ingredient: understand how and when the density becomes **non-linear** and collapses into **halos**.

Spherical collapse:



The sphere evolves as a closed universe:

$$\frac{3}{R^2}\left(\frac{dR}{dt}\right)^2 = \rho_m(t) - \frac{k}{R^2}$$

$$\rho_m \propto R^{-3}$$

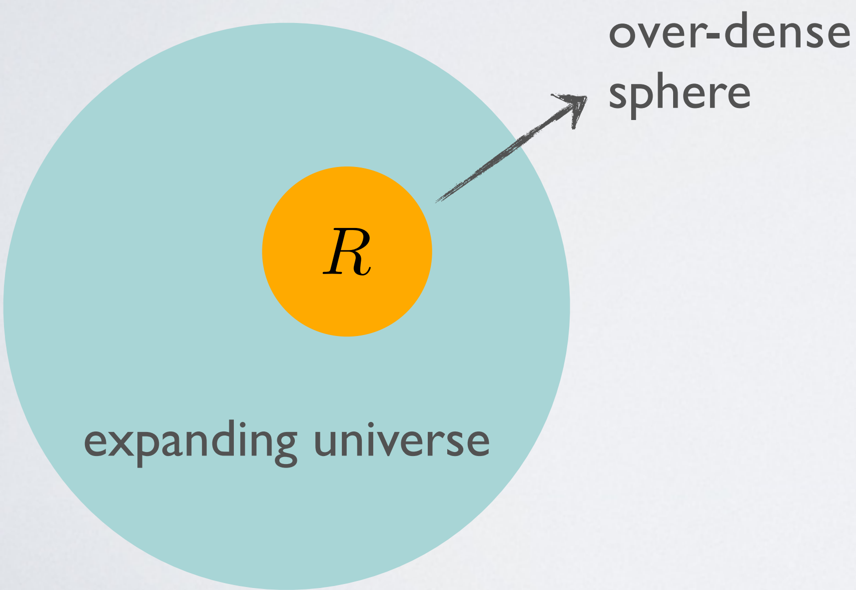
$$R(\eta) \sim 1 - \cos(A\eta)$$
$$t(\eta) \sim A\eta - \sin(A\eta)$$

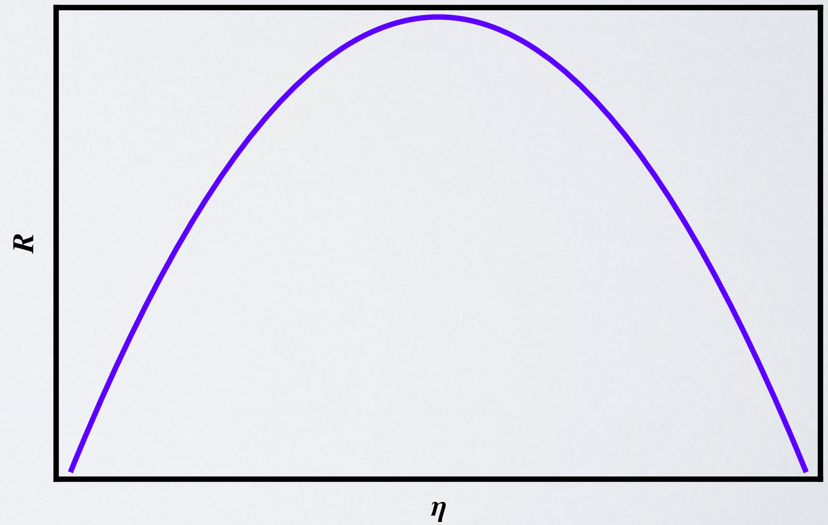
Spherical collapse

First ingredient: understand how and when the density becomes **non-linear** and collapses into **halos**.

Spherical collapse:

$$R(\eta) \sim 1 - \cos(A\eta)$$
$$t(\eta) \sim A\eta - \sin(A\eta)$$





Critical density

We calculate the **density contrast** $\delta_{\rm sph} = \dfrac{\rho_{\rm sph} - \bar{\rho}}{\bar{\rho}}$

In a matter dominated universe, the linear density contrast is:

$$\delta_{\rm sph} = \frac{3}{20}\left(\frac{6\pi t}{t_{\rm max}}\right)^{2/3}$$ with $t_{\rm max}$ time at $R_{\rm max}$

Collapse: $\delta_{\rm sph}(t_{\rm coll}) = \delta_{\rm sph}(2t_{\rm max}) \simeq 1.69$ $\qquad \delta^{\rm NL}_{\rm sph}(t_{\rm coll}) \simeq 180$

Two lessons:

- When the linear density contrast is larger than δ_c , the over-density **collapses**.
- A halo has an **average non-linear density** 200 larger than the background.

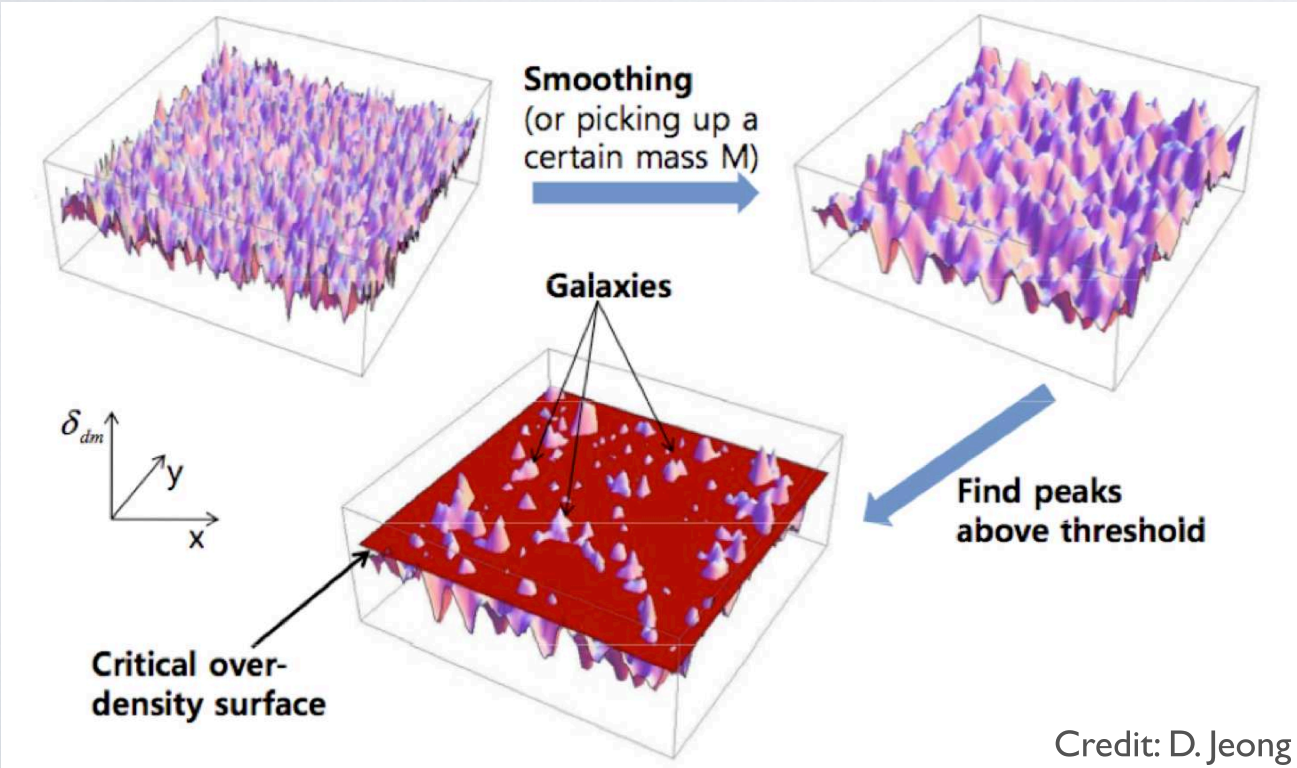
Distribution of halos

How can we predict the **distribution** of **halos** from the spherical collapse?

Let us **smooth** the linear density at a scale R

$$\delta_R(\mathbf{x}) = \int d^3\mathbf{x}'\, W(\mathbf{x}-\mathbf{x}', R)\, \delta(\mathbf{x}')$$

When $\delta_R > \delta_c$: collapse.



Above the **threshold** we have halos of initial radius R, i.e. mass

$$M \simeq \frac{4\pi R^3 \rho}{3}$$

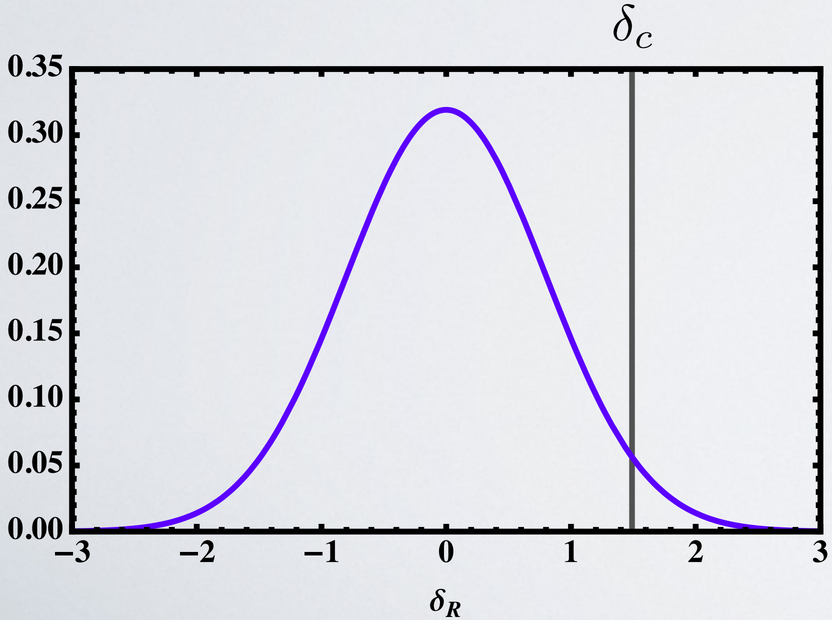
The **peaks** belong to halos of mass equal or greater than M .

Distribution of halos

We can calculate how many **peaks** we have.

- δ is gaussianly distributed with known variance.
- δ_R is also a **gaussian** with smooth variance σ_R

$$f(>M) = \int_{\delta_c}^{\infty} d\delta_R \, \mathcal{P}(\delta_R)$$



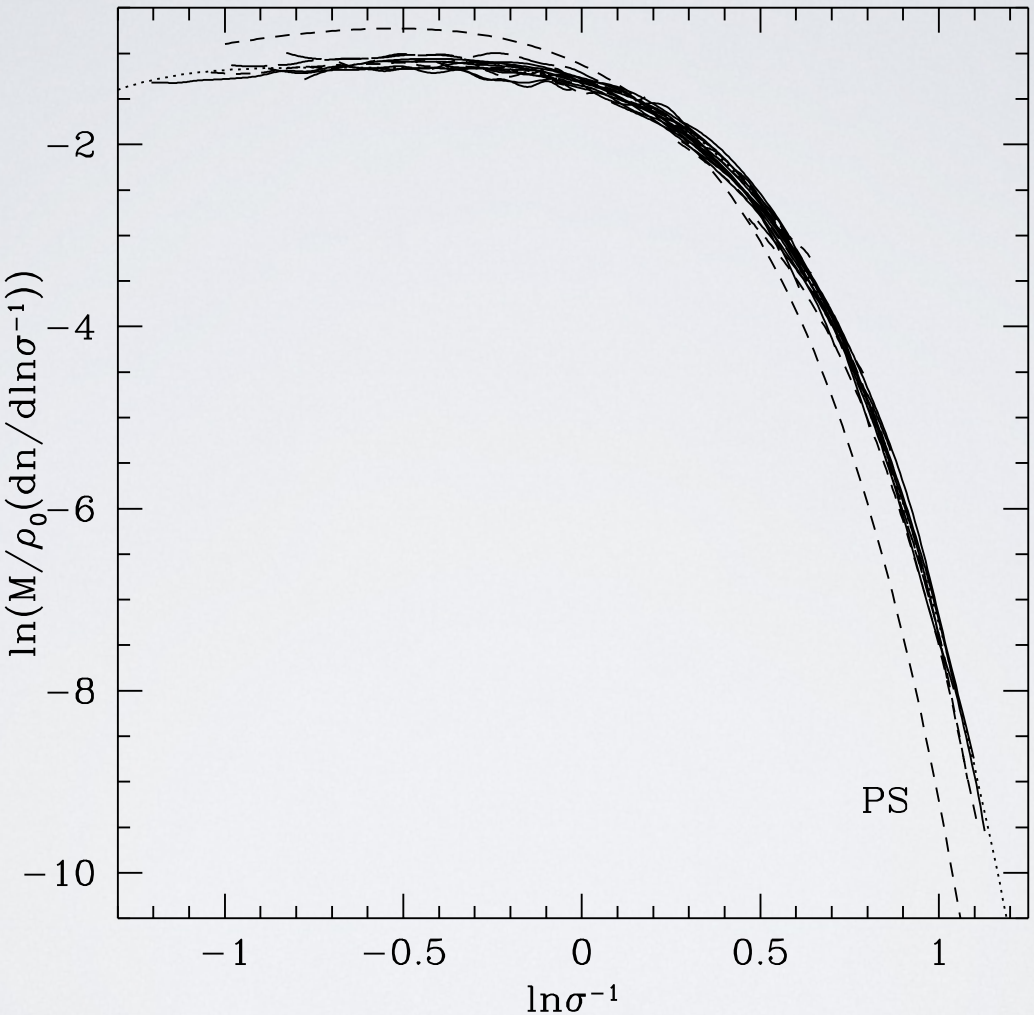
Repeat for all R, i.e. M and find the fraction of mass in halos of mass M

$$f(M)dM = f(>M+dM) - f(>M)$$

→ **Halo mass function**

Problem with **under-dense** region: x2

Halo mass function

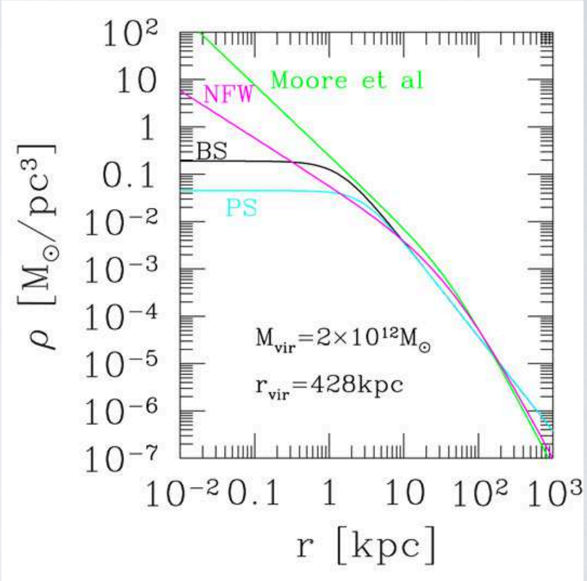


Credit: Jenkins et al. (2001).

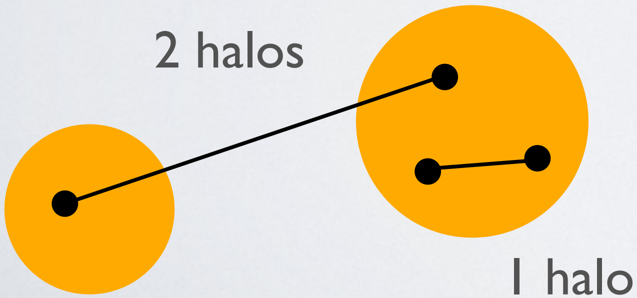
Density profile

The second ingredient is the **distribution** of mass **within** each **halo**.

- It is difficult to predict this analytically.
- Profile taken from numerical simulations.



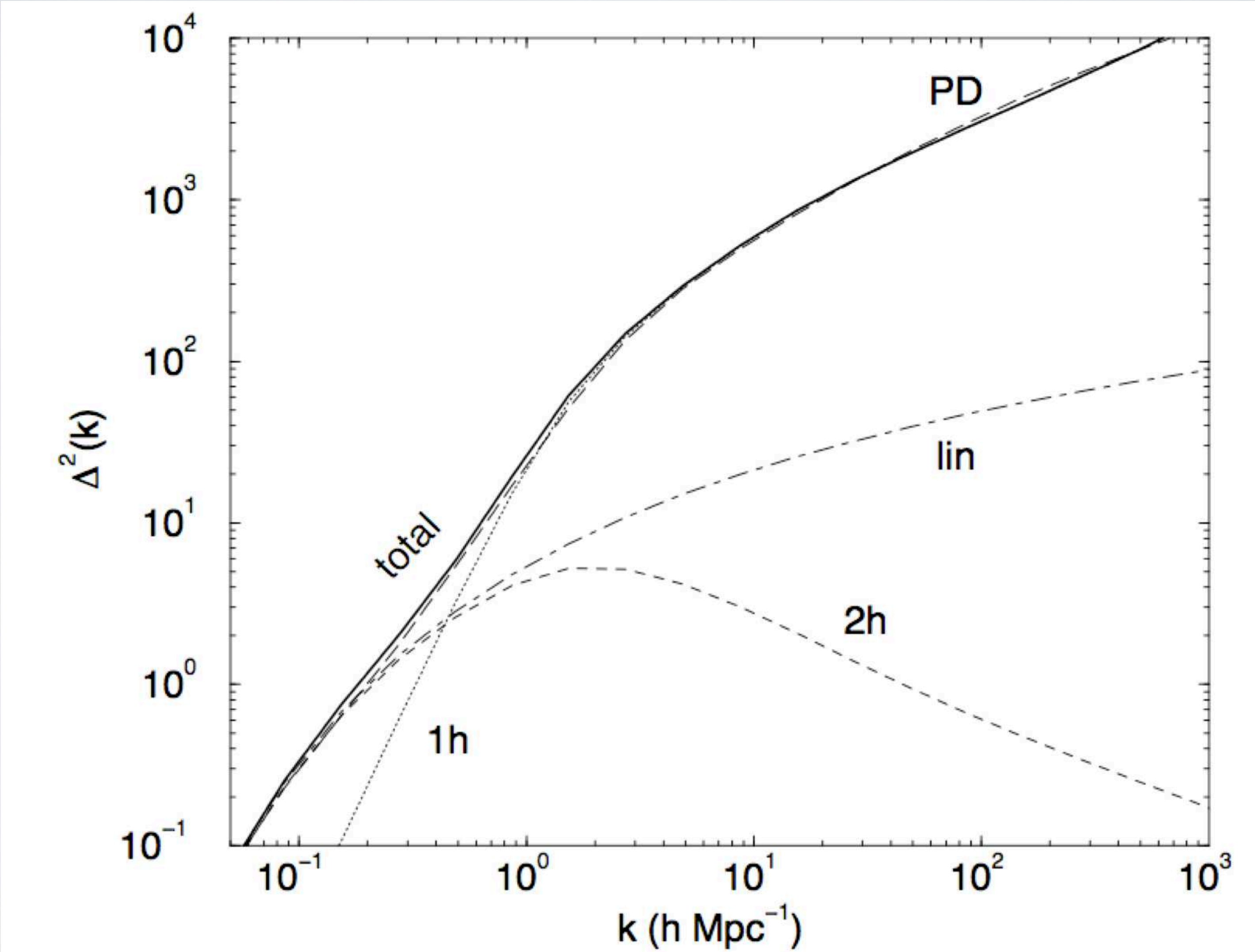
Enough to calculate the **correlation function**.



$$\xi(\mathbf{x},\mathbf{x}') = \xi^{1h}(\mathbf{x},\mathbf{x}') + \xi^{2h}(\mathbf{x},\mathbf{x}')$$

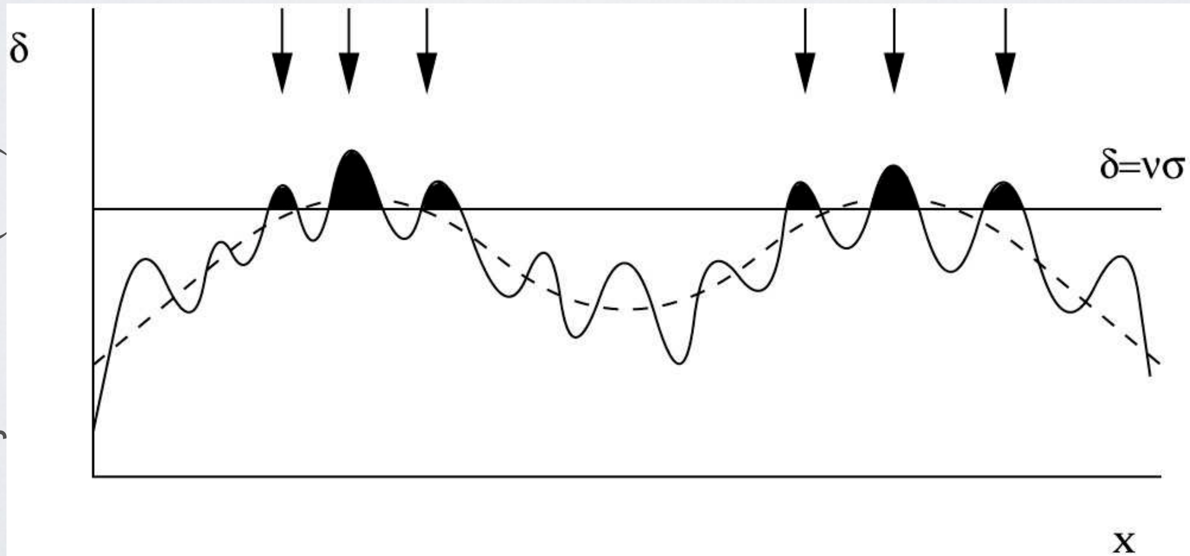
Power spectrum

Credit: Cooray and Sheth (2002)



Bias

- Consequence: galaxies are **biased tracers** of the dark matter.
- Galaxies form in halos: in a first approximation we can populate each halo with one galaxy.
- Halos form if the density is **higher** than the **threshold**.
- If we have a **long wavelength** mode of dark matter, it modulates the formation of halos.



Credit: J.A. Peacock (2003)